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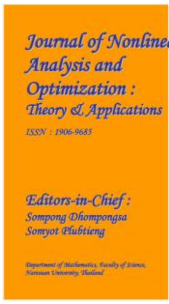


# A NOVEL PERSPECTIVE ON FUZZY SETS

**Dr. Rahul Deo Awasthi** Guest Faculty, Department of Mathematics Maharaja Chhatrasal Bundelkhand University, Chhatapur, Madhya Pradesh

## Abstract

In this paper, the authors offer a fresh perspective on Zadeh's concept of fuzzy sets. The central idea is that fuzzy sets are rooted in language and are inherently 'linguistic entities,' fundamentally distinct from 'crisp sets,' which originate from either physical collections of objects or explicit lists.

A new definition of fuzzy sets is proposed, incorporating two key magnitudes: a qualitative component, represented by a graph as the foundational element, and a quantitative component, expressed as a scalar magnitude. The graphs reflects the relational basis of fuzzy sets within language, while the scalar magnitude enabled by the 'measurement of word meanings' captures the numerical, extensional form in which the fuzzy set currently exists.

Given the importance of the scalar magnitude in practical applications, the concept of a 'working fuzzy set' is introduced. This incorporates the numerical function, the measure of meaning, or the membership function. The working fuzzy set broadens the scope of the original fuzzy set, allowing the same fuzzy set to manifest through different membership functions. In essence, the same graph can represent various extensional states. In contrast, a 'working crisp set' remains identical to the original crisp set.

**Keywords:**
*Fuzzy Set Theory, Fuzzy Logic, Meaning.*

## 1. INTRODUCTION

Since its introduction in 1965 by Lotfi A. Zadeh [8], a fuzzy set $f_P$ within a universe of discourse X has been characterized through its membership function, formally represented as the triplet $(X,P,\mu_P) = f_P$. Here, X denotes the universe of discourse, P refers to a predicate or property exhibited by elements in X, and $\mu_P$ is the membership function, where $\mu_P(x)$ indicates the degree, ranging from 0 to 1, to which an element xxx satisfies property P. When $\mu_P(x)$ takes values strictly of 0 or 1, P is considered rigid or crisp on X; conversely, values between 0 and 1 signify that PPP is vague, imprecise, or fuzzy. According to this definition, two fuzzy sets $f_P=(X, P, \mu_P)$ and $f_Q=(Y, Q, \mu_Q)$ are identical if and only if X=Y, P=Q, and $\mu_P = \mu_Q$. However, this is somewhat surprising, as practitioners know that the same fuzzy set can exhibit different membership functions, much like the same die can produce varying probability distributions depending on the context. Moreover, predicates P and Q do not need to be identical for their fuzzy sets to coincide it is sufficient for P and Q to be synonymous within X.

These points to a dual issue in Zadeh's original definition, regarding both the identity of fuzzy sets and the non-uniqueness of membership functions. The discrepancy arises from the differing foundational structures between probability theory, grounded in Boolean algebra, and language, to which *P* and $f_P$. Unlike Boolean algebra's rigid, commutative structure, language operates under more flexible, less formal rules [3, 5, 6].

Fuzzy sets have linguistic counterparts generated through common discourse to group entities sharing properties. For example, if X represents London's population and P= young the term "Young Londoners" describes those recognized as young to varying degrees. Similarly, "Old Londoners" and "Middle-Aged Londoners" represent other fuzzy collectives.

The innate human tendency to collectivize properties, whether rigid or gradual, underscores the linguistic origins of fuzzy sets. While collections like the set of weak Spanish vowels {i,e}can be explicitly listed, collectives such as "Young Londoners" cannot be confined to discrete elements, as the property of youth exists on a continuum.
This highlights the conceptual gap in Zadeh's definition, necessitating recognition of fuzzy sets as linguistic entities, shaped by commonsense reasoning essentially, language in practice [6]. A preliminary linguistic perspective on fuzzy sets was introduced by the first author in [4], though it conflated maximalist and measurability. This paper aims to bridge that gap by mathematically clarifying the nature of fuzzy sets, the role of membership functions, and distinguishing between theoretical constructs and their practical applications.

## 2. FUZZY SETS NEWLY DEFINED

**2.1.**To know how a predicate P linguistically acts on X, that is, how the elements in X can be distinguished by how they verify P, or how the property p varies along the universe of discourse, it should be known when, given two whatsoever elements x, y in X, which one of them shows p less than the other. That is, knowing if it is either x is less p than y, or y is less p than x.
Such variation that can be called the primary use on X of p, or of P, and when both possibilities happen, is said that x and y are equally P, are not distinguishable under P: The meaning of the statements x is P and y is P do coincide.
Shortening the statement x is less P than y by $x \prec_P y$, the (usually empirical) linguistic relation $\prec_P \subseteq X \times X$, facilitates the graph, or basic magnitude, $(X, \prec_P) = P$, the fuzzy set in X with linguistic label P.
Notice that this new definition induces a sensible change: Two fuzzy sets with respective linguistic labels P and Q are coincidental, provided both are in the same universe of discourse and are primarily used in the same form. That is:

$$\boldsymbol{P = Q} \Leftrightarrow (X, \prec_P) = (Y, \prec_Q) \Leftrightarrow X = Y \text{ and } \prec_P = \prec_Q .$$

The equality of two fuzzy sets means that their linguistic labels do have the same primary use or, in Wittgenstein words [7], (primary) meaning.
This solves the first problem with Zadeh's original definition of a fuzzy set, and offers a definition of P is contained in Q:

$$P \subseteq Q \Leftrightarrow \prec_P \subseteq \prec_Q,$$

Supposed X= Y.
Notice that with this new definition of fuzzy set it is not presumed that for being P = Q the predicates P and Q do be the same. To have the same meaning is to be synonyms.
Now lets try to solve the second: What is a membership function?
**2.2.** Since the idea behind membership functions coming from supposing that meaning has extensio is to know up to which numerical level it can be said that x is P, that is, measuring the degree up to which x is P, its extension.
Lets introduce when a function $m_P : X \to [0, 1]$ is a measure in the graph $(X, \prec_P)$. Such a function measures (is a measure of) the meaning of P in X, whenever the following three properties are satisfied:
(i) $x \prec_P y \Rightarrow m_P(x) \leq m_P(y)$, that is, the measure grows along the relation $\prec_P$ .
(ii) If z is minimal in the graph, that is, there is no t in X such that $t \prec_P z$, then $m_P(z) = 0$. Minimals do measure the minimum possible value. Notice that if an element has zero measure, it does not imply that it is a minimal; condition (ii) is necessary but not sufficient. Minimal elements are also called anti-prototypes of P in X.
(iii) If w is maximal in the graph, that is, there is no v in X such that $w \prec_P v$, then $m_P(w) = 1$. Maximals do measure the maximum possible value.
Notice that if an element measures one, it does not imply that it will be maximal; condition (iii) is necessary but not sufficient. Maximal elements are also called prototypes of P in X.
**2.3.** It is interesting to note that, with the characteristic function:

$$R(x, y) = \begin{cases} 1, & \text{if } x \prec_P y \\ 0, & \text{if } x \not\prec_P y \end{cases}$$

condition (i) is equivalent to:

$$\min(m_P(x), R(x, y)) \leq m_P(y). \quad (1)$$

This condition generalized to any fuzzy relation R, and in particular to fuzzy preorders, any membership function μ and using a continuous t-norm

$$T: T(\mu(x), R(x, y)) \leq \mu(y),$$

gives rise to the definition of the fuzzy logic states of a fuzzy relation R, studied in [3] and [1]. Actually, condition given by (1) establishes that $m_P$ and R satisfy the Modus Ponens Inequality with T ≤ min; so, condition (i) can also be seen as requiring that the measure $m_P$ of the predicate P given by property p, is logically consistent with relation ≺P.

The minimal and maximal elements of ≺P do not impose any restrictions using (1), however the minimum and maximum, if they exist, must have the smallest and largest value in $m_P$, respectively as it is imposed by laws (ii) and (iii) of $m_P$.

In addition, considering the relation $J_{min}$ defined by residua ion from min, (1) is equivalent to (see [3]):

$$R(x, y) < J_{min}(m_P(x), m_P(y)) = 1, \text{ if } m_P(x) \prec m_P(y);$$

or

$$J_{min}(m_P(x), m_P(y)) = m_P(y), \text{ in the contrary}$$

showing that $J_{min}$ is the maximum possible function R, that corresponds to the form and classical interpretation of x ≺P y, as not x is P, or y is P.

It should be noticed that, usually, axioms (i) to (iii) are not sufficient to specify a single measure; they just characterize all measures, but, in general, to design a single one more conditions are necessary. This is not rare at all; remember how many probabilities can be associated to the six faces of the same die. At such respect, let's present in the next paragraph some very simple examples.

Let's still notice that measures $m_P$ of P can be immediately identified with Zadeh's membership functions; in part, by clarifying what, concerning extensional meaning, appears in [8]. Thus, what is it a membership function is now mathematically clarified.

**2.3. Examples**

First

Let it be P = big in X = [0, 10]. Obviously, it is ≺big=≤, the total order of the Real Line (4 ≺big 6 ⇔ 4 ≤ 6). Hence the measures of the primary meaning of big in [0, 10] are the mappings $m_{big}$: [0, 10] → [0, 1] that, non-decreasing, verify the two border conditions $m_{big}(0) = 0$ and $m_{big}(10) = 1$, since in [0, 10] the only minimal is 0, the minimum, and the only maximal is 10, the maximum.

There is a great amount of these mappings and, without more conditions; it is not possible to specify one of them.

For instance, knowing that the measure it lineal, $m_{big}(x) = ax + b$, since the border conditions imply b = 0, and 10a = 1 ⇔ a = 1/10, it is clear that, for big in the closed interval [0, 10], there is just the unique lineal measure $m_{big}(x) = x/10$.

The situation is different when knowing that the measure is quadratic, $m_{big}(x) = ax2 + bx + c$. In this case the border conditions imply c = 0 and 100a + 10b = 1, or b = (1−100a) 10 = 0.110a, with which it is $m_{big}(x) = ax2 + (0.110a)\, x$, giving a one-parameter family of quadratic functions that, with a = 0, just recovers the lineal measure, and with a = 0.01 gives its square $m_{big}(x) = (x\ 10)\ 2$ . Second.

a) If X = [0, 10], and P = "greater than four", it is obvious that P = {x ∈ [0, 10]; 4 ≺ x} = (4, 10]. In this case P is crisp and, consequently, P is a set, a collection of numbers, a semi-closed interval.

b) Returning to the first example for a while, let's consider that "big" is equated to "greater than eight". It is $P = \{x \in [0, 10];\ 8 \prec x\} = (8, 10]$, and their membership function is

$$m_P(x) = \{0, \text{ if } 0 \leq x \leq 8$$
$$\{1, \text{ if } 8 < x \leq 10,$$

a non-decreasing function verifying m(0) = 0 and m(10) = 1; clearly, is a particular case of "big".

Notice that in both the linear and the quadratic measures of "big", the membership functions or measures are continuous, but in this case there is a discontinuity of the measure at point x = 8. The crisp character of P in X causes the breaking of the measure

**2.4.** Let's consider what happens when P is rigid in the universe X, a case in which the binary linguistic relation ≺P reduces to "x is equally P than y".

For instance, in the set N of Natural Numbers with P = "odd", it is clear that 5 is equally odd as 55 but not that 66; all odd numbers are equally, totally, odd, and the rest of the numbers, the pairs, are not odd at all. Something with a remarkable difference from the Londoners in for instance, the "Middle Age" collective [2].

By writing $=P=\prec_P \cap \prec_p^{-1}$, the fuzzy set of odd numbers results to be the graph $(X, =_{odd})$ that, being an algebraic equivalence, classifies perfectly X in two equivalence classes, that of odd numbers and that containing all the other numbers.

Consequently, since a number whatsoever n is either odd, or is not odd at all, there is just the following measure of the meaning of "odd":

$$m_{odd}(x) = \begin{cases} 1 & x \text{ is odd} \\ 0 & x \text{ is not odd} \end{cases}$$

and the set $m^{-1}_{odd}(1)$ is exactly the set rigid predicate odd specifies in N, accordingly with the so-called "axiom of specification" in the Nave Theory of Sets.

## 3. WORKING FUZZY SETS

**3.1.** What we did is just passing from a basic magnitude, the graph, or fuzzy set $(X, \prec_P) = P$, up to a scalar magnitude $(X, \prec_P, m_P)$ reflecting the state in which the predicate is currently managed and allowed to change of the relation $\prec_P$ by the (new) relation

$$x \prec^m_p y \Leftrightarrow m_P(x) \le m_P(y),$$

a binary relation that not only is a new one, but is also different from $\prec P$ since, for instance, the graph $(X, \prec^m_P)$ is a total or lineal one, because of all pair of points $(x, y) \in X \times X$, one of them will have smaller or equal measure than the other, that is, it will be necessarily either $x \prec m\ P\ y$, or $y \prec^m_P x$. Thus, under $\prec^m_P$ there are not incomparable, orthogonal, elements in X; but, instead, under $\prec P$ such elements can exist, since it can be perceptively impossible capturing if one of both x and y is less P than the other.

**3.2.** The new graph $(X, \prec^m_P) = P^m$, is called a "working fuzzy set associated to P" since, usually, practitioners don't work with $\prec^P$ but with the total order relation $\le$ of the Real Line inherited by the unit interval [0, 1] that with the measure is the former total binary relation we just introduced.

Obviously, if P is rigid on X, the fuzzy set reduces to a set that, in addition, and since in this case there is just a unique measure, coincides with the corresponding working fuzzy set.

It should be noticed that the working fuzzy set $P^m$ is, in general, larger than the fuzzy set P. In fact:

$$x \prec P\ y \Rightarrow m_P(x) \le m_P(y) \Leftrightarrow x \prec^m_P y,$$

that is, $\prec P \subseteq \prec^m_P$.

It can be said that the act of measuring the meaning enlarges the basic linguistic relation between the elements in the universe of discourse. Hence, the practitioner should be cautious at the respect since not all result valid in Pm will be, necessarily, neither valid in P, nor in a different working fuzzy set, one endowed with a different measure.

Notice that in the first case of the former example 1, it is

$$x \prec_{\text{big}} y \Leftrightarrow x \le y \Leftrightarrow \frac{x}{10} \le \frac{y}{10} \Leftrightarrow m_{\text{big}}(x) \le m_{\text{big}}(y) \Leftrightarrow x \prec^m_{\text{big}} y,$$

thus, the original linguistic relation $\prec P$ and the new (measure dependant) one $\prec m\ P$ are coincidental: A coincidence that is not general.

## 4. DERIVATE PREDICATES

**4.1.** In natural language, graded concepts that are usually represented by fuzzy sets are characterized by admitting various related predicates such as the antonym or predicates intensified or weakened by modifier operators. The use of those predicates in the language is of course related to the use made of the main or primary predicate.

For example, the use of the antonym predicate $ant_P$ could correspond to the inversion of the order that defines the use of *P*:

$$x \prec_{antP} y \Leftrightarrow y \prec_P x,$$

and a measure $m_{antP}$ should satisfy

$$x \prec_{antP} y \Leftrightarrow y \prec_P x \Rightarrow m_{antP}(x) \le m_{antP}(y) \wedge m_P(x) \ge m_P(y)$$

that is, it should invert the order associated with the extension of P. The minimal elements for P will

become maximal and vice versa
We can easily get an extension simply by defining

$$m_{antP}(x) = 1 - m_P(x),$$

and the usual negation in fuzzy sets allows us to obtain a valid extension for the antonym that, nevertheless, makes it coincidental with the negation "not P". However, it is usual that in graded predicates the antonym does not coincide with the negation but rather that the antonym implies the negation of the predicate, that is

If x is ant*P*, then x is not P,

without always holding the reciprocal.
Another way to define the antonym is to take a bijection s on X that reverses the order defined by P

$$x \prec_P y \Leftrightarrow s(y) \prec_P s(x).$$

The membership function defined as

$$m_{antP}(x) = m_P(s(x)),$$

is a valid extension for $ant_P$, since

$$x \prec_{antP} y \Leftrightarrow y \prec_P x \Leftrightarrow s(x) \prec_P s(y) \text{ then } m_P(s(x)) \prec_P m_P(s(y)) \Leftrightarrow m_{antP}(x) \prec m_{antP}(y).$$

**4.2.** In example 2.5 the indistinguishability decomposes into a preorder R and its reciprocal Rr (x, y) = R(y, x). The antonym predicate can be constructed using the reciprocal predicate by taking the minimum element column of the universe X = [0, 10], which is a maximal element for the reciprocal predicate,

$$R^r(0, y) = R(y, 0) = 10 - y,$$

which is the linear function decreasing from the points (0, 1) to (10, 0). That membership function is also obtained by means of the inner bijection s(x) = 10 − x on X from the extension of P = "big" given by the column of the maximum 10 of the preorder R

## 5. CONCLUSION

A fuzzy set P can be considered "empty" when its linguistic label refers to nothing within the universe X. This occurs either when the binary linguistic relation ≺P\prec P≺P is unknown or explicitly known to be empty. In such situations, PPP is described as metaphysical in X. Conversely, P is deemed measurable in X if it holds some definable measure or meaning.
It is important to highlight that in the metaphysical case, no membership function can be determined. However, the question arises: if a membership function is null, can we conclude that the fuzzy set is empty? The fact that every element in X has a membership degree of zero does not imply the nonexistence of the graph of PPP, the relation ≺P\prec P≺P, or the fuzzy set itself. The only definitive conclusion is that PPP is empty if and only if ≺P\prec P≺P is the empty set, in which case any measure becomes meaningless.
Still, if the fuzzy set has a null measure, it cannot have a prototype or a maximal element in the universe, as no element has a positive degree of membership. Is it valid to claim that PPP is empty in this scenario? It is reasonable to define PPP as empty if and only if ≺P\prec P≺P is empty or the measure is null.
This issue has persisted in fuzzy set literature since its inception, as noted in [8], stemming from defining fuzzy sets through membership functions by analogy with classical, crisp sets, treated mathematically independent of linguistic considerations, even with the axiom of specification included.
Similarly, the condition under which a fuzzy set PPP is equivalent to the universe XXX is straightforward: when, for all x∈Xx \in Xx∈X, the statement " xxx is PPP" holds absolutely. In this case, the membership function mP(x)=1m_P(x) = 1mP(x)=1 for all x∈Xx \in Xx∈X, the relation ≺P\prec P≺P equals X×X \times X×X, and every element xxx in X serves as a proto type of P.

## REFERENCES

1. A.R. de Soto, E. Trillas, and S. Cubillo, *Classes of fuzzy sets with the same material conditional*, International Journal of Approximate Reasoning, **16**(3-4), 225-233 (1997).
2. P.R. Halmos, *Naive Set Theory*, Dover Books (1960).
3. E. Trillas, *On the Logos: A Naïve View on Ordinary Reasoning and Fuzzy Logic*, Springer (2017).

4. E. Trillas, *On the Use of Words and Fuzzy Sets*, Information Sciences, **176**(11), 1463-1487 (2003).
5. E. Trillas and L. Eciolaza, *Fuzzy Logic: An Introductory Course for Engineering Students*, Springer (2015).
6. E. Trillas, S. Termini, and M.E. Tabacchi, *Reasoning and Language at Work: A Critical Essay*, Springer (2022).
7. L. Wittgenstein, *Philosophical Investigations*, Basil Blackwell (1953).
8. L.A. Zadeh, *Fuzzy Sets*, Information and Control, **8**(3), 338-353 (1965).